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On Percolation with Fibers or Layers

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We consider site percolation on Z^d, directed edges going from any $s \in Z^d$ to $s+A_1,..., s+A_n$, where $A_1,..., A_n$ are the same for all sites and at least two of them are noncollinear. A site is closed if it belongs to $p+\text{Block}$, where p is a point in a Poisson distribution in $R^d \supset Z^d$ with a density θ and $\text{Block} = \{s \in L : |s| \leqslant M\} + \{s \in R^d : |s| \leqslant \rho\}$, where L is a linear subspace of R^d, $|\cdot|$ is the Euclidean norm, $\rho = \max(|A_1|,..., |A_n|)$ and M is a parameter. We study the behavior of θ^*, the critical value, and P^*_{closed}, corresponding critical percentage of closed sites, when $M \to \infty$. Denote R^d/L the factor space. Call two nonzero vectors U, V codirected if $U = kV$, where $k > 0$. **Theorem.** If there are A_i and A_j whose projections to R^d/L are not codirected, then $\theta^* \asymp 1/M^{\dim(L)}$ and P^*_{closed} remains separated both from 0 and 1 when $M \to \infty$. If projections of all $A_1,..., A_n$ to R^d/L are codirected, then $\theta^* \asymp 1/M^{\dim(L)+1}$ and $P^*_{\text{closed}} \asymp 1/M$ when $M \to \infty$.

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It is often assumed in percolation theory that states of sites (or bonds) are mutually independent. When percolation is used to describe properties of materials, this assumption implies chaotic structure. However, some real materials contain fibers or layers, which seems to imply long-distance correlations along certain directions combined with no or much shorter correlations across these directions. To take this possibility into account, we study asymptotics of critical values in oriented site percolation on a graph $\mathscr{A}$, which has Z^d as the set of sites, with a special distribution of closed vs. open sites. Given several non-zero *arrows* $A_1,..., A_n \in Z^d$, at least two of which are non-collinear, directed edges of the graph $\mathscr{A}$ go from any $s \in Z^d$ to $s+A_1,..., s+A_n$. Each site is either closed or open according to the following random rule. We embed Z^d into R^d with the same axes and

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use the Euclidean norm $|\cdot|$ and distance. Denote $\text{Ball} = \{s \in R^d : |s| \leqslant 1\}$ the unit ball in R^d. Given a linear subspace L of R^d and a positive parameter M, the set $\text{Block} \subset R^d$ is defined as the vector sum

$$\text{Block} = M \cdot (\text{Ball} \cap L) + \rho \cdot \text{Ball} \tag{1}$$

where ρ is a large enough constant, say $\rho = \max(|A_1|,..., |A_n|)$. We define the distribution of closed sites by the rule: A site is closed if and only if it belongs to $p + \text{Block}$, where p is a point of a Poisson point distribution in R^d with a density θ. We call points of this distribution Poisson points. As usual, the probability that there are k Poisson points in a measurable set $S \subset R^d$ equals $e^{-\theta \cdot \text{vol}(S)} \cdot (\theta \cdot \text{vol}(S))^k/k!$, where $\text{vol}(S)$ is the volume (i.e., measure) of S.

Call a sequence of sites $s_1,..., s_k$, where $k \geqslant 1$, a path from s_1 to s_k if for every $i = 1,..., k-1$ there is an edge from s_i to s_{i+1}. A path is called open if all its sites are open. A site t is called reachable from site s if there is an open path from s to t. We say that there is percolation from site s to ∞ if the set of sites reachable from s is infinite. Since the probability of percolation to ∞ is one and the same from all sites, we call it the percolation probability and assume that percolation always starts at the origin. Also we denote $P_{\text{closed}}(\theta)$ the percentage of closed sites. Since

$$P_{\text{closed}} = 1 - e^{-\theta \cdot \text{vol(Block)}} \tag{2}$$

P_{closed} depends on θ in the monotonic way, but the percolation probability depends on θ in the anti-monotonic way, it cannot increase when θ increases. Accordingly we denote θ^* the infimum of those values of θ, for which the percolation probability equals zero and $P^*_{\text{closed}} = P_{\text{closed}}(\theta^*)$. It is easy to prove that $0 < \theta^* < \infty$ and $0 < P^*_{\text{closed}} < 1$ in all the situations considered here.

Our theorem describes the asymptotic behavior of θ^* and P^*_{closed} when $M \to \infty$, the arrows and L remaining the same. Values which do not depend on M (but depend on arrows and L) are called constants or parameters. For any functions F, $G > 0$, $F \asymp G$ means that $\text{const} \cdot G \leqslant F \leqslant \text{const} \cdot G$, where both constants are positive. Denote $\dim(\cdot)$ dimension. Denote R^d/L the factor space. Call two non-zero vectors U, V codirected if $U = kV$, where $k > 0$. Let us say that the "along" case takes place if there are two arrows, whose projections to R^d/L are not codirected. The other case, namely when projections of all the arrows to R^d/L are codirected, is called the "across" case. For example, if we take d arrows equal to the unit vectors directed along the axes of Z^d and define L by an equation $k_1 x_1 + \cdots + k_d x_d = 0$, where $x_1,..., x_d$ are the coordinates, then we have the "across" case if all $k_1,\ldots, k_d$ are positive or all of them are negative and the "along" case otherwise. Our main result is the following:

Theorem. In the "along" case $\theta^* \asymp 1/M^{\dim(L)}$ and P^*_{closed} remains separated both from 0 and 1 when $M \to \infty$. In the "across" case $\theta^* \asymp 1/M^{\dim(L)+1}$ and $P^*_{\text{closed}} \asymp 1/M$ when $M \to \infty$.

Since $\text{vol}(\text{Block}) \asymp M^{\dim(L)}$, (2) allows to deduce our estimations for P^*_{closed} from those for θ^*, so it remains only to estimate θ^*.

With every site we associate a variable, which equals 1 if this site is open and 0 if it is closed. This allows us to use the following well-known partial order between normed measures on $\{0, 1\}^S$, where S is any countable set (see, e.g., in 1). Given two configurations $x, y \in \{0, 1\}^S$, we write $x \prec y$ or $y \succ x$ if $x_i \leqslant y_i$ for all $i \in S$. We call a real function f on $\{0, 1\}^S$ monotonic if $x \prec y \Rightarrow f(x) \leqslant f(y)$. For any normed measures μ, ν on $\{0, 1\}^S$ we write $\mu \prec \nu$ or $\nu \succ \mu$ if $\int f\, d\mu \leqslant \int f\, d\nu$ for any monotonic f.

We denote $\text{lin}(\cdot)$ the linear hull and $H = \text{lin}(A_1,\ldots, A_n)$ the linear hull of the arrows. Of course, percolation from the origin depends only on the states of sites in H. Let us explain how we can substitute the distribution of open vs. closed sites defined above by another one, defined only on $Z^d \cap H$. Let us denote $\mu(M, \rho, \theta)$ the distribution on $\{0, 1\}^{Z^d \cap H}$, corresponding to the distribution defined above. Now let us define another distribution $\mu'(M', \rho', \theta')$ on $\{0, 1\}^{Z^d \cap H}$ as follows: we denote

$$\text{Block}' = M' \cdot (\text{Ball} \cap L \cap V) + \rho' \cdot (\text{Ball} \cap V) \tag{3}$$

and declare a site in $Z^d \cap H$ closed if it belongs to $(p' + \text{Block}')$, where p' is a point of a Poisson distribution in H with a density θ'. Then there are positive constants $C_1,\ldots, C_6$ such that

$$\mu(M, \rho, \theta) \prec \mu'(C_1 \cdot M, C_2 \cdot \rho, C_3 \cdot \theta \cdot M^{\dim(L) - \dim(L \cap H)}) \tag{4}$$

and

$$(M, \rho, \theta) \succ \mu'(C_1 \cdot M, C_5 \cdot \rho, C_6 \cdot \theta \cdot M^{\dim(L) - \dim(L \cap H)}) \tag{5}$$

for all M, ρ and small enough θ. Let us explain why (4) and (5) are true. Let us call a point $p \in R^d$ relevant if $p + \text{Block}$ intersects H.

First, let us reduce the general case to the case $\text{lin}(L, H) = R^d$. If $\text{lin}(L, H) \neq R^d$, then all the relevant points are at a distance $\leqslant$const from $\text{lin}(L, H)$, so we can substitute them by their orthogonal projections to $\text{lin}(L, H)$ and thereby substitute the initial distribution on $Z^d \cap \text{lin}(L, H)$

by another one. If we denote these distributions $\nu(M, \rho, \theta)$ and $\nu'(M, \rho, \theta)$, it is easy to prove that

$$\nu(\text{const} \cdot M, \text{const} \cdot \rho, \text{const} \cdot \theta) \prec \nu'(M, \rho, \theta) \prec \nu(\text{const} \cdot M, \text{const} \cdot \rho, \text{const} \cdot \theta)$$

with suitable positive constants.

Now let $\text{lin}(L, H) = R^d$. Then any shift of L intersects H, so for any p we can call its projection $\text{proj}(p)$ that point in $(p + L) \cap H$, which is the nearest to p. Then, if we substitute every relevant Poisson point by its projection, we obtain a Poisson distribution on H with a density $\theta' = \text{const} \cdot \theta \cdot M^{\dim(L) - \dim(L \cap H)}$. Also let us denote $\text{dist}(p)$ the distance from p to H.

Now, to prove (4), it is sufficient to observe that there is a positive constant C_{far} such that if $\text{dist}(p) > C_{\text{far}} \cdot M$ then p is irrelevant and if $\text{dist}(p) \leqslant C_{\text{far}} \cdot M$ then $(p + \text{Block}) \cap H$ is covered by $\text{proj}(p) + \text{Block}'$ where Block' is defined by (3) with $M' = \text{const} \cdot M$ and $\rho' = \text{const} \cdot \rho$ with large enough constants.

To prove (5), observe that there is a positive constant C_{near} such that if $\text{dist}(p) \leqslant C_{\text{near}} \cdot M$, then the intersection $(p + \text{Block}) \cap H$ includes $\text{proj}(p) + \text{Block}'$ where Block' is defined by (3) with $M' = \text{const} \cdot M$ and $\rho' = \text{const} \cdot \rho$ with small enough positive constants.

Upper Estimation in the "Along" Case. (The assumption that this is the "along" case is not actually used here.) Let us assume that $\theta \cdot M^{\dim(L)}$ exceeds a large enough parameter λ and prove that the probability of percolation equals 0 for all $M \geqslant \text{const}$. Choose a basis $B_1, \dots, B_{\dim(L)}$ in L and complement it by $B_{\dim(L)+1}, \dots, B_d$ to obtain a basis $B_1, \dots, B_d$ in R^d. For every $y = (y_1, \dots, y_d) \in Z^d$ we define

$$\text{cloud}(y) = \alpha \left(M \cdot \sum_{i=1}^{\dim(L)} y_i \cdot B_i + \sum_{i=\dim(L)+1}^{d} y_i \cdot B_i \right) + \tfrac{1}{2} \cdot \text{Block}$$

where α is a positive parameter. We call these sets clouds and choose α so small that the union of clouds covers R^d. We call a cloud open if it contains at least one open site and closed otherwise. Notice that if a cloud contains a Poisson point p, it is completely covered by $p + \text{Block}$ and therefore closed. Also notice that there is a constant C_0 such that no point belongs to more that C_0 different clouds. Since volumes of all clouds equal $\text{vol}(1/2 \cdot \text{Block})$, which we denote V_0, the volume of a union of r different clouds is not less than $V_0 \cdot r/C_0$. Since $V_0 \asymp M^{\dim(L)}$, the probability that r different clouds are open does not exceed $e^{-\theta \cdot V_0 \cdot r/C_0} \leqslant (e^{-\text{const} \cdot \lambda})^r$, where the constant is positive. Now suppose that there is percolation, that is there is an infinite open path starting at the origin. Taking a long enough piece of this path and substituting every site in this path by some cloud containing it, we get a sequence of open clouds. If two terms of this sequence coincide, we delete one of them and all the terms between them. Repeating this deleting while it is possible, we obtain another sequence of open clouds, in which all the terms are different. For any cloud in this sequence the next cloud can be chosen only out of a constant of different ones. Thus the probability that there is a sequence of this sort does not exceed $(\text{const} \cdot e^{-\text{const} \cdot \lambda})^r$, where r is the number of terms, which can be made arbitrarily large. This expression tends to zero when $r \to \infty$ provided λ is large enough. ∎

Upper Estimation in the "Across" Case. Let us assume that $\theta \cdot M^{\dim(L)+1}$ exceeds a large enough parameter λ and prove that the probability of percolation equals 0 for all $M \geqslant \text{const}$. The inequality (4) allows us to concentrate on H, having a site closed if it belongs to $p + \text{Block}'$, where Block′ is defined by (3) and p is a point of a Poisson distribution on H with the density $\theta' = \text{const} \cdot \theta \cdot M^{\dim(L) - \dim(L \cap H)}$. Now we call points of this distribution Poisson points. Since projections of all the arrows to R^d/L are codirected, $\dim(L \cap H) = \dim(H) - 1$. Let us choose any basis $B_1, ..., B_{\dim(H)-1}$ in $L \cap H$ and then denote $B_{\dim(H)}$ that normed vector orthogonal to $L \cap H$, whose projection to R^d/L is codirected with projections of all the arrows. Thus $B_1, ..., B_{\dim(H)}$ is a basis in H. For every $y = (y_1, ..., y_{\dim(H)}) \in Z^{\dim(H)}$ we define:

$$\left.\begin{aligned} &\text{center}(y) = \alpha \cdot M' \cdot \sum_{i=1}^{\dim(H)} (y_i \cdot B_i) \\ &\text{cloud}(y) = \text{center}(y) + \tfrac{1}{8} \cdot \text{Block}' + \bigcup_{0 \leqslant x \leqslant \beta} (x \cdot M' \cdot B_{\dim(H)}) \\ &\text{screen}(y) = \text{cloud}(y) + 2\beta \cdot M' \cdot B_{\dim(H)} \end{aligned}\right\} \tag{6}$$

where α and β are positive parameters to be chosen later. Let us call a site free if there is an open path from this site to ∞ and call a cloud free if it contains at least one free site. Let us explain why, choosing the constant β small enough, we can assure that if a screen contains at least one Poisson point, then the corresponding cloud cannot be free. Any $p \in H$ can be decomposed into a sum $p = l(p) + m(p) \cdot B_{\dim(H)}$, where $l(p) \in L \cup H$ and $m(p) \in R$. Notice that $m(A_i) > 0$ for all A_i and denote

$$m_0 = \max_i m(A_i) \qquad \text{and} \qquad T = \max_i \frac{|l(A_i)|}{m(A_i)}$$

Now assume that M' is large enough, that there is a Poisson point $p \in \text{screen}(y)$ and at the same time there is an infinite open path $s_0, s_1, s_2, ...$

starting at some site $s_0 \in \text{cloud}(y)$ and come to a contradiction. From (6) $m(p) \geqslant m(s_0)$. As we go along our path, the value of $m(s_i)$ strictly increases at every step at most by m_0. So there is i such that

$$|m(s_i) - m(p)| \leqslant m_0 \tag{7}$$

whence

$$\begin{aligned} |l(s_i) - l(p)| &\leqslant |l(s_0) - l(p)| + |l(s_i) - l(s_0)| \\ &\leqslant |l(s_0) - l(p)| + T \cdot |m(s_i) - m(s_0)| \end{aligned}$$

Also note that $|l(s_0) - l(p)| \leqslant M'/2$. Since

$$m(s_i) - m(s_0) \leqslant m_0 + m(p) - m(s_0) \leqslant m_0 + 3\beta M'$$

we conclude that $l(s_i) - l(s_0) \leqslant T \cdot (m_0 + 3\beta M')$. Therefore

$$|l(s_i) - l(p)| \leqslant |l(s_0) - (l(p)| + |l(s_i) - l(s_0)| \leqslant M'/2 + T \cdot (m_0 + 3\beta M')$$

We can choose $\beta > 0$ so small that the last expression is less than M'. This and (7) are sufficient to have $s_i \in p + \text{Block}'$, which provides the contradiction we sought. After that we choose α so small that the union of clouds covers H. Let us denote V_0 the volume of a screen and note that $V_0 \asymp M^{\dim(H)}$. Since there is a constant C_0 such that no point belongs to more than C_0 different screens, the volume of a union of r different screens is not less than $\text{const} \cdot V_0 \cdot r/C_0$. Since $\theta' \cdot M^{\dim(H)} > \text{const} \cdot \lambda$, the probability that r different clouds are free does not exceed $e^{-\text{const} \cdot \theta' \cdot V_0 \cdot r/C_0} \leqslant (e^{-\text{const} \cdot \lambda})^r$ where all the constants are positive. Then we argue like in the "along" case, only speak of free, rather than open, clouds.

Now we go to the lower estimations. Denote Q the graph with the set of vertices $Z^2 = \{(x, y) : x, y \in Z\}$ in which two oriented bonds go from every vertex (x, y) to $(x+1, y)$ and $(x, y+1)$. We shall use the following fact[2, 1] formulated here as a lemma:

Lemma. There is $\varepsilon > 0$, e.g. $\varepsilon = 1/20$, such that the probability of oriented site percolation in Q from the origin to ∞ is positive provided the following: for any r sites the probability that all of them are closed does not exceed ε^r.

Lower Estimation in the "Along" Case. Let us assume that $\theta \cdot M^{\dim(L)}$ is less than a small enough positive parameter λ and prove that the probability of percolation in $\mathscr{A}$ from the origin to ∞ is positive for all $M \geqslant \text{const}$. We choose two non-collinear arrows, say A_1 and A_2, whose projections to R^d/L are not codirected, reduce our attention to the subgraph of $\mathscr{A}$ generated by them, and prove that the probability of percolation for M large enough is positive even on this subgraph. This subgraph is isomorphic with Q, which allows us to use the lemma. Due to (5), we may assume that the Poisson distribution is given only on this subgraph, which we now call $\mathscr{A}$, and that $d=2$ and there are only two arrows A_1 and A_2 from the very beginning. Therefore the only possible values for $\dim(L)$ are 0, 1 and 2. The case $\dim(L)=0$ is easy, actually it is the "classical" case without long-range correlations. Thus we consider three cases.

Case (a). $\dim(L)=1$ and both A_1 and A_2 do not belong to L. Since the projections of A_1 and A_2 to R^2/L are not codirected, they have opposite signs. We set B_1 equal to that normed vector in L, which can be represented as a linear combination of A_1 and A_2 with positive coefficients. Then we define $\text{center}(y)=\alpha\cdot(M\cdot y_1\cdot B_1+y_2\cdot A_2)$ for all $y=(y_1, y_2)\in Z^2$.

Case (b). $\dim(L)=1$ and A_1 belongs to L, but A_2 does not. In this case $\text{center}(y)=\alpha\cdot(M\cdot y_1\cdot A_1+y_2\cdot A_2)$ for all $y=(y_1, y_2)\in Z^2$.

Case (c). $\dim(L)=2$. In this case $\text{center}(y)=\alpha\cdot M(y_1\cdot A_1+y_2\cdot A_2)$ for all $y=(y_1, y_2)\in Z^2$.

In all these cases we define $\text{cloud}(y)=\text{center}(y)+\beta\cdot\text{Block}$ and $\text{screen}(y)=\text{center}(y)+2\beta\cdot\text{Block}$. Screens are defined so that if a cloud is closed, the corresponding screen contains at least one Poisson point.

For any $p\in R^2$ we denote $[p]\in Z^2$ that point, whose coordinates are integer parts of the respective coordinates of p. In each case we form an oriented graph G with Z^2 as the set of vertices, in which an oriented bond goes from y to y' if there is a path in $\mathscr{A}$ from $[\text{center}(y)]$ to $[\text{center}(y')]$, all the terms of which belong to $\text{cloud}(y)\cup\text{cloud}(y')$. We declare a cloud and the corresponding site of G open if all the sites of $\mathscr{A}$ belonging to this cloud are open; otherwise both are closed. Due to these definitions, if there is percolation in G from the origin to ∞, then there is percolation in $\mathscr{A}$ from the origin to ∞ also.

In all the three cases we choose first α and then β so large that the clouds cover all R^2 and at least two bonds go from any vertex $y=(y_1, y_2)$ of G to (y_1+1, y_2) and (y_1, y_2+1), and in result G has a subgraph isomorphic with Q.

Since the area of any screen is $\asymp M$, the probability that a screen contains at least one Poisson point does not exceed $1-e^{-\text{const}\cdot\theta M}\leqslant 1-e^{-\text{const}\cdot\lambda}$. Since there is a constant C_0 such that no point belongs to more than C_0 screens, for any r screens the probability that each of them contains at least one Poisson point does not exceed ϕ^r, where $\phi=(1-e^{-\text{const}\cdot\lambda})^{1/C_0}$. Therefore for any r clouds the probability that all of them are closed also does not exceed ϕ^r. Hence, from the lemma, the probability of percolation in G from the origin to ∞ is positive as soon as $\phi < 1/20$, which is true for λ small enough. Therefore the probability of percolation in $\mathscr{A}$ also is positive. ▮

Lower Estimation in the "Across" Case. Here we assume that $\theta \cdot M^{\dim(L)+1}$ is less than a small enough positive parameter λ and prove that the probability of percolation in $\mathscr{A}$ from the origin to ∞ is positive for all $M \geqslant \text{const}$. In this case we choose any two non-collinear arrows, call them A_1 and A_2, reduce attention to the subgraph generated by them and prove that the probability of percolation is positive even there for M large enough. The argument is practically the same as in the case (c) of the "along" case. ▮

Note. Percolation is sometimes used to model destruction of materials. In this connection the statement of our theorem in the "across" case presents a theoretical possibility of greater robustness than one might expect: the material as a whole does not collapse even when almost all (i.e., all except const/M) of its elements are destroyed. This phenomenon is more general than that particular distribution which is used here. There are other ways to define a distribution of closed sites (or closed bonds) with strong correlations along a certain subspace L and weak correlations across it, in which the asymptotic behavior of P^*_{closed} is the same as stated here. one of these ways is to use the same definition (1), but let a site open (rather than closed) if it belongs to $(p + \text{Block})$, where p is a point in a Poisson distribution in $R^d \supset Z^d$ with a density θ. (In this case both the percolation probability and P_{closed} depend on θ in the monotonic way: they cannot decrease when θincreases.) Then all the statements of our theorem remain true with one exception: in the "across" case $\theta^* \asymp \ln M / M^{\dim(L)}$.

A similar phenomenon has been described for the multiscale non-oriented percolation; see the chapter "Multiscale Percolation Schemes" in the survey 3.

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